Math 118B Final Practice

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1. In the compact metric space X a sequence of functions (f_n)—not necessarily continuous—converge pointwise to a continuous function f. Prove that the convergence is uniform if and only if for any convergent sequence $x_n \to x$ in X we have
$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

2. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(2\pi\sqrt{n^2+x^2}\right)$ converges uniformly on bounded intervals.

3. Prove that the series $\displaystyle\sum_{n=1}^{\infty} n^2x^2e^{-n^2|x|}$ converges uniformly on $\mathbb{R}$.

4. Determine the domain of convergence for $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{(-1)^n n^2} x^n$.

5. Suppose that $f, g : [0,1] \to \mathbb{R}$ are continuous. Prove there exists $c \in [0,1]$ so that
$$\int_0^1 f(x)g(x)\,dx = f(c)\int_0^1 g(x)\,dx.$$

6. Fix $0 < a < b$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$. Evaluate
$$\lim_{\epsilon\to 0^+}\int_{a\epsilon}^{b\epsilon}\frac{f(x)}{x}\,dx.$$

7. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and $f \geq 0$. If
$$\int_0^1 f(x)\,dx = 0,$$
prove that f is identically 0.

8. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and that for each $n \geq 0$,
$$\int_0^1 x^n f(x)\,dx = 0.$$
Prove that f is identically 0.

9. Prove that $\displaystyle\int_0^{\infty}\sin(x^2)\,dx$ converges.

10. Evaluate the limits

    (a) $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$

    (b) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(\ln k)^2 - \left(\frac{1}{n}\sum_{k=1}^{n}\ln k\right)^2\right)$

11. Define the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$$

Prove the following:

(a) The function f is discontinuous at every point.

(b) The function f is not Riemann integrable on any bounded interval.

12. Define the Riemann ruler function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/q & \text{if } x = p/q \text{ with } p, q \text{ relatively prime integers} \end{cases}$$

Prove the following:

(a) The function f is continuous at the irrationals and continuous at the rationals.

(b) The function f is Riemann integrable on every bounded interval.

(c) For any $a < b$ we have

$$\int_a^b f(x)\, dx = 0.$$

13. A continuous function $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies $|K(x,y)| < 1$ for all $(x,y) \in [0,1] \times [0,1]$. Prove there is a unique continuous $f : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) + \int_0^1 K(x,y) f(y)\, dy = e^x.$$

14. Let (f_n) be a sequence of real-valued uniformly bounded equicontinuous functions on a metric space X. If we define

$$g_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\},$$

prove that the sequence (g_n) converges uniformly.

15. Suppose that K is a nonempty compact subset of a metric space X. Given $x \in X$ prove there exists a point $z \in K$ so that

$$d(x,z) = \operatorname{dist}(x, K).$$

16. Prove every compact metric space has a countable dense subset.

17. Evaluate $\displaystyle\lim_{n\to\infty} \int_0^1 \left(1 + \frac{x}{n}\right)^n dx$ with justification.

18. Show that

$$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \cdots$$

converges uniformly $\mathbb{R}$ but never absolutely.

19. (a) Prove the polynomials of even degree are dense in the space of continuous functions $C[0,1]$.

(b) Is this still true on $C[-1,1]$?

20. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable on every interval of the form $[0, A]$ for $A > 0$ and that $f \to 1$ as $x \to \infty$. Prove that

$$\lim_{s\to 0^+} s \int_0^\infty e^{-st} f(t)\, dt = 1.$$

21. Define, for $x, y > 1$

$$f(x,y) = \frac{x-y}{1-xy}.$$

For each fixed y, note that $f(x,y) \to 1$ as $x \to 1$. Is the convergence uniform in y?

22. (a) Suppose that (a_{nk}) is a doubly-indexed series of positive terms. Prove that

$$\sum_k \sum_n a_{nk} = \sum_n \sum_k a_{nk},$$

where ∞ is allowed.

(b) Give an example of a sequence for which the above equation fails.

23. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Define $T : C[0,1] \to C[0,1]$ to be the linear operator

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Prove that T maps bounded subsets of $C[0,1]$ into precompact ones.

24. (a) Let f be a continuous periodic function with some period t. Show that its set of translates

$$\mathcal{F} = \{f(x-t) : t \in \mathbb{R}\}$$

is compact in $C(\mathbb{R})$.

(b) A function is called almost periodic if its set of translates is precompact. Prove the set of almost periodic functions is a closed subalgebra of $C(\mathbb{R})$.